Input–Output Dynamics as an Iterated Function System

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Abstract.

We reconsider the input–output model in the light of a new mathematical formalism, namely the so called Iterated Function Systems (IFS), that, in spite of the linearity of the structure of the system, allows a greatly complex dynamical behavior. The basic theory is presented and some analytical results are provided, together with some simulations (at least partly) based on empirical data.

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1. Introduction.

Consider a generic input-output model (T, δ_t), $t = 1, 2, \ldots$, where T is an $n \times n$ matrix giving production coefficients for each of the n goods in the economy and the δ_t are n-dimensional vectors giving final demands for goods at time $t = 1, 2, \ldots$. Under the assumption of an uniform, unitary production lag for the various goods, the dynamics of such a model can be expressed by the linear difference vector equation[1]:

$$\mathbf{x}_{t+1} = T\mathbf{x}_t + \delta_t \tag{1}$$

where $\mathbf{x}_t$ is the n-dimensional vector which gives the production of the n goods at time t. Solving (1) by recurrence, it is apparent that the dynamical behavior of our input-output system depends on the eigenvalues of the T matrix and that the range of possible behavior displayed by the system is quite limited. To see this, consider:

$$\delta_\infty = \lim_{n \to \infty} \sum_{i=0}^{n-1} T^{n-i-1} \delta_i \tag{2}$$

and assume that it exists and is finite. Then one out of three structurally stable cases occurs: either (1) is (cyclically or monotonously) globally asymptotically stable w.r.t. δ_∞, or (1) is (cyclically or monotonously) globally asymptotically unstable w.r.t. δ_∞, or δ_∞ displays saddle instability. The dynamical character of (1) entirely depends on the eigenvalues of T, and more specifically on whether their real part is in absolute value less than one. Cyclical behavior is in principle also possible but

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[1]Here we do not consider capital goods.

it is structurally unstable, i.e., it is destroyed by small perturbations of parameter values.

To sum up, a traditional linear input-output model can hardly give rise to a realistic dynamic behavior, such as persistent cyclical behavior that is apparently one of the typical stylized patterns which are observed in real economic systems.

On the other hand, there are several good reasons to use input-output models in the description of a production economy. First, these models keep the ad-hocness of the model specification to a minimum, because they can be interpreted as a linear approximation of the true and perhaps unknowable structure of the economy, provided that the linearized dynamics has no eigenvalues with unitary modulus[2]. This feature of the input-output model is also desirable because it allows estimation and testing by means of standard econometric techniques. Second, the linear approximation argument captures the important fact that the production technology is basically fixed in each given period, and that this brings about a major constraint on the production choices in the very short run. This can be seen by observing that the linear approximation described by (1) has an essentially local nature, i.e. it presumably holds only in a small neighborhood of the linearization point[3].

The question then arises whether it is possible to reconcile the highly desirable characteristics of classical input-output models with the necessity of using models which allow for more complex dynamic behavior than that displayed by linear models. The aim of this paper is to show that this is actually the case.

The plan of the rest of the paper is the following. Section 2 provides a discussion of deterministic vs. stochastic dynamic aggregate models,

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[2]This is a consequence of the Hartman–Grobman theorem; see Guckenheimer and Holmes (1983), p. 18.

[3]In the case where the linearization is recursively made around the last value of the production vector $\mathbf{x}_t$, the T matrix would also become time dependent. In this case, by imposing some regularity conditions on the nature of the time dependence of the technical coefficients, the solution of (1) would take the form $\mathbf{x}_t = Z(t, t_0)[x_0 + \int_{t_0}^{t} Z^{-1}(s, t_0)\delta(s)ds]$ where $Z(t, t_0) = e^{\int_{t_0}^{t} T(s)ds}$ [see Burton (1985), sec. 1.1].

with special reference to input-output dynamics. Section 3 gives some basic facts ragarding IFSs. Section 4 intepprets the results of section 3 in terms of input-output dynamics. Section 5 discusses some properties of the equilibrium dynamics of the model linked to the topological structure of its attractor. Section 6 presents some simulations. Section 7 concludes.

2. Input–output dynamics as a bridge between linear stochastic dynamics and complex deterministic dynamics.

Recent research in nonlinear dynamics has shown that suitably chosen, deterministic nonlinear models are able to generate random–looking time series that classical statistical techniques are not able to distinguish from those generated by stochastic models in which the basic deterministic structure is perturbed by random shocks. These results have sparkled a debate on whether one should use deterministic nonlinear models rather than linear models with random shocks to model aggregate fluctuations [see Brock (1991) and Perli and Sandri (1991) for up to date surveys]. No conclusive answer to the question has so far been reached. We show in this paper how a fair compromise between the two position may be reached under the form of an input-output dynamics with shocks.

In the standard specification of a linear dynamic macro model with shocks, the basic deterministic structure of the model is perturbed by an additive random shock ϵ_t; in the context of a dynamic input-output system, one has

$$\mathbf{x}_{t+1} = T\mathbf{x}_t + \delta_t + \epsilon_t \tag{3}$$

Although widely used, this specification is unnecessarily restrictive. There is in principle no particular reason to believe that random shocks act additively on an otherwise deterministic structure. An important source of variability may in principle be generated by random shocks acting on the *structural* part of the model itself. In particular, there is no reason to rule out the existence of shocks of a specifically technological nature that directly affect factor productivity. If $T \equiv [a^{ij}]$, we may therefore write:

$$T[\zeta_t] \equiv [a^{ij}(\zeta_t^{ij})] \tag{4}$$

In its most general form, our dynamic input–output model with shocks can therefore be written

$$\mathbf{x}_{t+1} = T[\zeta_t]\mathbf{x}_t + \delta_t + \epsilon_t \tag{5}$$

Let $h(\zeta_t, \epsilon_t)$ be the joint density (probability) of the additive demand shock and of the multiplicative technology shock; assume for simplicity that h is stationary, i.e., the t subscripts may be dropped. Let $(\zeta^\theta, \epsilon^\theta)$ belong to the support of h, $\theta \in \Theta$ being the index of the support. A density function (probability distribution) over $(T^\theta, \delta^\theta)$, $\theta \in \Theta$, is therefore defined, where $T^\theta \equiv T[\zeta^\theta]$, $\delta_t^\theta \equiv \delta_t + \epsilon^\theta$. The dynamics (5) takes therefore the form

$$\mathbf{x}_{t+1} = T^\theta \mathbf{x}_t + \delta_t^\theta \qquad \text{with density (probability) } p_\theta,\ \theta \in \Theta \tag{6}$$

In other words, a stochastic dynamics may always be reinterpreted as a family of deterministic dynamics that are selected from time to time on the basis of a given density (probability distribution). This interpretation of the model nicely fits into an important class of dynamical systems recently introduced by Barnsley (1988) and known as iterated function systems (IFS).

3. Iterated Function Systems.

As already pointed out in section 1, it is well known that linear dynamical systems display only extremely simple kinds of behavior. It has therefore long been believed that, in order to produce complex deterministic behavior, and most notably chaotic behavior, one would have to work with nonlinear structures. Barnsley (1988) has shown that this is not actually the case, at least as long as we do not confine our attention to dynamical systems defined by single maps and consider the dynamics generated by a *family* of maps. In this latter case, it is relatively easy to show that, under certain conditions, even families of linear maps may produce deterministically complex behavior, in a sense to be specified later on. The dynamics generated by a family of maps may be seen either as a set valued dynamics defined as the union of the images of the various maps of the family or, equivalently, as an IFS with probabilities where each map of the family is activated at every given time with a given probability; thus, even if the framework is basically deterministic, there is here a natural way to model the action of random factors. Moreover, the ancient prejudice about the limitations of linear systems as a credible tool for the modeling of realistic dynamic phenomena seems therefore bound to vanish, at least as long as one admits a certain degree of variability in the underlying structure that justifies the use of a family of maps to represent the data generating process rather than of a single one. As we have shown in section 2, this kind of assumption is rather natural in the context of a dynamic aggregate input-output model; it is consistent with the Frisch- Slutzky approach to dynamic modelling that emphasizes the importance of exogenous shocks [see e.g. Lines (1990)] and also with the approach that emphasizes the importance of endogenous factors in the explanation of the variability of the observed time series [see e.g. Long-Plosser (1983)]. In our perspective there is therefore no conflict between the two approaches, but rather a mutual enforcement. In the remainder of this section we introduce some definitions and some basic results regarding IFSs. In the next section, we show how our dynamic input-output model can be nicely interpreted as an IFS.

A transformation $w : \Re^n \rightarrow \Re^n$ of the form:

$$w(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, \tag{7}$$

where A is an $n \times n$ real matrix and $\mathbf{b}$ is a real column vector, is called *affine* transformation.

A transformation $f : \mathbf{X} \rightarrow \mathbf{X}$ on a metric space $(\mathbf{X}, d)$ (where $d = ||\cdot||$ is a vector norm) is called *contraction* mapping if there is a constant $0 \leq s < 1$ (the contractivity factor[4]) such that:

$$d(f(\mathbf{x}), f(\mathbf{y})) \leq s\, d(\mathbf{x}, \mathbf{y}) \qquad \forall\ \mathbf{x}, \mathbf{y} \in \mathbf{X}. \tag{8}$$

[4]The contractivity factor of an affine transformation is also known as the *matrix norm* induced by the vector norm $d = ||\cdot||$ and is defined

Given a metric space $(\mathbf{X}, d)$[5], let $(\mathcal{H}(\mathbf{X}), h_d)$ denote the corresponding metric space of nonempty compact subsets of $\mathbf{X}$ (other then the empty set), with the Hausdorff metric[6] h_d. Then a *hyperbolic Iterated Function System* (IFS henceforth) consists of a finite set of contraction mappings $w_n : \mathbf{X} \to \mathbf{X}$ (with respective contractivity factors $s_n \in [0, 1)$) for $n = 1, 2, \ldots, N$. Moreover we call:

$$s = \max\{s_n : n = 1, 2, \ldots, N\} \tag{9}$$

the IFS contractivity factor.

The following theorem states a first important result: the family of affine contractive maps is a contractive transformation itself and it has a unique globally stable attractor given by the union of the attractor of the single maps.

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by:

$$s = |||A||| = \max_{||\mathbf{x}||=1} ||A\mathbf{x}||.$$

If $||\mathbf{x}||_\infty = \max_{1\leq i,j\leq n} |x_i|$ (l_∞ vector norm), then $s = |||A|||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|$ (maximum row sum matrix norm);
if $||\mathbf{x}||_1 = \sum_{i=1}^n |x_i|$, then $s = |||A|||_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$ (maximum column sum matrix norm);
if $||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^n x_i^2}$ (Euclidean norm), then $s = |||A|||_2 = \max\{\sqrt{\lambda} : \lambda$ is an eigenvalue of $A^T A\}$ (spectral norm), where A^T is the transpose of A.
For further details see, for example, Horn and Johnson (1985).

[5]Without apparent losses of generality, from now on we take d to be the standard Euclidean metric.

[6]If we define the distance from the point $\mathbf{x} \in \mathbf{X}$ to the set $B \in \mathcal{H}(\mathbf{X})$ as: $d(\mathbf{x}, B) = \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in B\}$, and the distance from the set $A \in \mathcal{H}(\mathbf{X})$ to the set $B \in \mathcal{H}(\mathbf{X})$ as: $d(A, B) = \max\{d(\mathbf{x}, B) : \mathbf{x} \in A\}$, then the Hausdorff distance between two sets A and B in $\mathcal{H}(\mathbf{X})$ is defined by:

$$h_d(A, B) = \max\{d(A, B), d(B, A)\}.$$

THEOREM 1. `[Barnsley (1988)]` *Let* $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$ *be a hyperbolic IFS on a metric space* $(\mathbf{X}, d)$ *with contractivity factor* s *as given in (9). Then the transformation* $W : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ *given by:*

$$W(B) = \bigcup_{n=1}^{N} w_n(B) \qquad \forall\ B \in \mathcal{H}(\mathbf{X}) \tag{10}$$

is a contraction mapping, defined on the metric space $(\mathcal{H}(\mathbf{X}), h_d)$*, with contractivity factor* s*; that is:*

$$h_d(W(B), W(C)) \leq s \cdot h_d(B, C) \qquad \forall\ B, C \in \mathcal{H}(\mathbf{X}). \tag{11}$$

Moreover the transformation W *has a unique fixed point* $F \in \mathcal{H}(\mathbf{X})$*, called the attractor of the IFS, which obeys* $F = W(F) = \cup_{n=1}^{N} w_n(F)$*, and which is given by:*

$$F = \lim_{n \to \infty} W^{(n)}(B) \qquad \forall\ B \in \mathcal{H}(\mathbf{X}), \tag{12}$$

where $W^{(n)}(B) = W \circ W \circ \cdots \circ W(B)$ n *times.*

The next theorem establishes the continuous dependence[7] of the attractor of an IFS on parameters:

THEOREM 2. `[Barnsley (1988)]` *Let* $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$ *be a hyperbolic IFS on a metric space* $(\mathbf{X}, d)$*, with contractivity factor* s*. Let* w_n*, for* $n = 1, 2, \ldots, N$*, depend continuously on* $p \in P$*, where* P *is a compact metric space. Then the attractor of the IFS* $F(p) \in \mathcal{H}(\mathbf{X})$ *depends continuously on* $p \in P$*, with respect to the Hausdorff metric* h_d*.*

Finally we report an interesting theorem which represent the only result at present available for the theoretical determination of the *Hausdorff dimension*[8] of the attractor of an IFS.

The singular value function $\phi^D(A)$ of an affine transformation $w = A\mathbf{x} + \mathbf{b}$ is defined by:

$$\phi^D(A) = \lambda_1 \cdots \lambda_{m-1} \lambda_m^{D-m+1} \qquad \text{for } 0 < D \leq n \tag{13}$$

[7]We say that $f_p = w(p, \cdot)$ depends continuously on $p \in P$ for each $\mathbf{x} \in \mathbf{X}$ if $f : P \to \mathbf{X}$ is continuous.

[8]Given a set $F \subset \Re^n$, we call δ-cover of F a family $\{U_i\}$ of non-empty subset of $\Re^n$ whose diameters $|U_i| = \sup\{||\mathbf{x} - \mathbf{y}|| : \mathbf{x}, \mathbf{y} \in U\}$ do not

where m is the smallest integer greater than or equal to D ($m-1 \leq D \leq m$) and λ_i, $i = 1, 2, \ldots n$ are the singular values of A (with the convention $1 > \lambda_1 > \ldots > \lambda_n > 0$), that is, if we adopt the Euclidean metric, the positive square roots of the eigenvalues of the matrix $A^T A$.

THEOREM 3. [Falconer (1988)] *Assume that* $|||A_i||| < \frac{1}{3}$ *for* $1 \leq i \leq N$. *For almost all* $\beta = (\mathbf{b}_1, \ldots, \mathbf{b}_N) \in \Re^{nN}$ *(in the sense of nN-dimensional Lebesgue measure), we have:*

$$\dim F(\beta) = \min\{n, d(A_1, \ldots, A_N)\} \tag{14}$$

where $\dim F(\beta)$ *is the Hausdorff dimension of the attractor of the IFS,* $d(A_1, \ldots, A_N)$ *is defined as the unique* $D > 0$ *such that:*

$$\lim_{r\to\infty} \left[\sum_{\mathbf{i}\in J_r} \phi^D(A^{\mathbf{i}})\right]^{\frac{1}{r}} = 1 \tag{15}$$

given that $J_r = \{(i_1, \ldots, i_r) : 1 \leq i_j \leq N\}$ *is the set of all the sequences of length* $r \in \mathbf{N}$ *formed using the integer 1 to N, and* $A^{\mathbf{i}} = A_{i_1} \cdot A_{i_2} \cdots A_{i_r}$ *is the product of matrices indexed by* $\mathbf{i} \in J_r$.

Now we consider another important class of IFS: hyperbolic *IFSs with probabilities*. They consist of an IFS $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$ together

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exceed δ and such that $F \subset \cup_{i=1}^{\infty} U_i$. Then:

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\}$$

is the Hausdorff s-dimensional outer measure of F (where the infimum is over all δ-covers $\{U_i\}$ of F) and:

$$\dim F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

is the Hausdorff dimension of F. If in particular $s < \dim F$, one has $\mathcal{H}^s(F) = \infty$; whereas, for $s > \dim F$, $\mathcal{H}^s(F) = 0$. For $s = \dim F$, if $0 < \mathcal{H}^{\dim F}(F) < \infty$, F is said to be an s–set. By definition F has a fractal structure if its Hausdorff dimension s is greater than its topological dimension [see Mandelbrot (1982)].

with a set of positive real numbers $Q = \{p_1, p_2, \ldots, p_N\}$, such that $\sum_{i=1}^{N} p_i = 1$, where each p_i is associated with the transformation w_i. In other words, an IFS with probabilities is a family of contraction mappings which are 'activated' with probabilities p_i.

For our purposes it is convenient to call the two following theorems: the first states the existence of a unique invariant measure which 'lives' on the attractor of the IFS while the second represents an ergodic theorem for IFSs.

THEOREM 4. [Barnsley (1988)] *Let* $(\mathbf{X}, d)$ *be a compact metric space and let* $\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$ *be an hyperbolic IFS with probabilities. Then, there is a unique invariant measure*[9] μ *associated to the IFS whose support is the attractor* F*, given in (12) of the IFS* $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$.

THEOREM 5. [Elton (1987)] *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let* $\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$ *be an hyperbolic IFS with probabilities and let* $\{\mathbf{x}_n\}_{n=0}^{\infty}$ *denote an orbit of the IFS, that is:*

$$\mathbf{x}_n = w_{\sigma_n} \circ w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_1}(\mathbf{x}_0),$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n \in \{1, 2, \ldots, N\}$ *and are chosen independently according to the probabilities* $p_1, p_2, \ldots, p_N$*. Let* μ *be the unique invariant measure for the IFS. Then, with probability one:*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(\mathbf{x}_k) = \int_{\mathbf{X}} f(\mathbf{x}) d\mu(\mathbf{x}) \tag{16}$$

for all continuous functions $f : \mathbf{X} \to \Re$ *and all* $\mathbf{x}_0$.

Strictly speaking, IFSs are set dynamical systems, i.e., the dynamics acts on compact subsets of $\Re^n$ rather than on points. On the other

[9] A probability measure μ is called *invariant* if it has the following property: if μ is the probability measure over the state s_t in period t, then it is also the probability measure over the state s_{t+1} in period $t+1$. In other words, μ in invariant iff it is a fixed point of the operator Q, that is $\mu = Q\mu$, where Q is the operator that maps the set of probability measure on a measurable space $(S, \mathcal{L})$ into itself.

hand, IFSs with probabilities are dynamical systems in the familiar sense. Theorems 4 and 5 lay a bridge between them because they show that a) the invariant dynamics of an IFS with probability always lives on the attractor of the corresponding deterministic IFS; b) every point of the attractor of a deterministic IFS is visited almost surely by the invariant dynamics of the corresponding IFS with probabilities.

4. Input-output dynamics as an iterated function system.

For the sake of the interpretation of the IFSs formalism of section 3 in terms of a dynamic input-output model, one could think that the assumption that IFSs must have *finite* cardinality (i.e., that we consider only finite families of affine maps) is somewhat restrictive. In terms of our discussion in section 2, this means that we must assume that technology and demand shocks have only a finite number of realizations, i.e., that the structural variability of the model is described by a probability *distribution* rather than by a density. At a closer look, however, the assumption is justified, and even recommended, by the (often neglected) existence of technical indivisibilities. Although some research on (Countably) Infinite IFSs (IIFSs) is currently going on, it has therefore little interest for our purposes [see Brandt (1989), Fernau (1991), and Staiger (1991)].

With this qualification, and with the further qualification that a nonnegativity constraint on all coefficients is imposed to preserve economic meaningfulness[10], the generical affine map $w(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is just another way of writing our input-output system $T\mathbf{x} + \delta$.

Moreover we need to establish under what conditions our input-output models are contraction mappings. If we assume the Euclidean vector norm l_2, then the contractivity factor s equals the square root of the largest eigenvalue of the matrix $A^T A$. As the coefficients of A are all nonnegative and $A^T A$ is simmetric, the Perron-Frobenius theorem states that the largest eigenvalue $\lambda_{\max}$ of the matrix $A^T A$ is real and nonnegative, greater than the minimum row (or column) sum and lower

[10]Of course, we also impose a nonnegativity constraint on the set $\mathbf{X}$ of production vectors.

than the maximum row (or column) sum, that is:

$$\min_r\{\sum_i(a_{ir}\sum_j a_{ij})\} \leq s \leq \max_r\{\sum_i(a_{ir}\sum_j a_{ij})\} \tag{17}$$

Now, if $\sum_j a_{ij} \leq 1\ \forall i = 1, 2, \ldots, n$ then:

$$\sum_i(a_{ri}\sum_j a_{ij}) \leq \sum_i a_{ri} \leq 1 \qquad \forall\, r = 1, 2, \ldots, n.$$

Therefore the economic sustainability condition $\sum_j a_{ij} \leq 1\ \forall\, i = 1, 2, \ldots, n$, together with the above nonnegativity constraint on the production coefficients, implies $0 \leq s \leq 1$[11].

We can therefore interpret our input-output dynamics as an IFS $\{w_1, \ldots, w_N\}$ with probabilities $Q = \{p_1, \ldots, p_N\}$, where $N = card[\Theta]$. In view of the discussion of section 2, these probabilities are the probabilities of the joint realizations of technology and demand shocks. We therefore warn the reader not to think of different technologies T as of truly different technological options for the economy. Basically, the same technology is here adopted in all periods and the variability of production coefficients must be traced back to productivity shocks. Different realizations of the shocks correspond therefore to 'bad' or 'good' states according to whether or not factor productivities and/or final demands are at satisfactory levels.

The results on IFS dynamics reported in the previous section allow us to draw rather naturally some sharp conclusions: given Theorems 4 and 5 above, the probability distribution of the shocks does not have any bearing on the structure of the attractor of the dynamics, but only influences transitional behavior (more specifically, the frequence with which the economy visits a given region of the attractor). Moreover, according to Theorem 3, what ultimately determines the geometric structure of the attractor are the singular values of the technological matrices, while

[11]Otherwise, if we consider the vector norm l_1 (see note 4), s equals the maximum row sum of the coefficients of the matrix A. Again, the economic sustainability condition and the nonnegativity constraint on the production coefficients imply $0 \leq s \leq 1$.

in general the $\mathbf{b}_i$ vectors do not affect its fractal dimension. In terms of our input-output model, this theorem tells us that the 'degree of complexity' of the dynamics of the production vectors do not depend on the exogenous demand shocks (which, in turn, affect the variability of the production levels x_i, $i = 1, 2, \ldots, n$), but is endogenously determined by the technological coefficients. This latter result strengthens of course the link between our IFS input-output dynamics and a real business cycle interpretation of economic fluctuations. Moreover, the results of section 3 allow us to gain some insight regarding the interaction of endogenous and exogenous forces in the determination of aggregate variability. However, a given specification of the model is realistically tenable only in the short–medium run, because of technical change that acts on the relevant technology matrices T and therefore on the structure of the IFS. For this reason, transitional behavior substantially matters for welfare analysis; whether or not the resulting dynamics are satisfactory from a social planner's point of view depends on the historical realizations of the shocks. One way to model technical change in our context is to introduce shift parameters in the specification of the IFS as explained in section 3; notice in particular that from Theorem 2 above the attractor of an IFS displays continuous dependence on parameters and therefore one is sure that technological change does not alter the structure of the attractor substantially, at least as long as the change is not dramatic.

5. Invariant dynamics.

It is worth to point out that the existing mathematical analysis of the dynamics of IFSs is basically concerned with the characterization of the *invariant* dynamics, i.e., of the dynamics that take place on the attractor once all transients have died out. In other words, we only have results for the long- run behavior of our input-output systems. On the other hand, it turns out that the long-run dynamics are generally quite 'complicated' and therefore an explicit analysis of the adjustment dynamics cannot add much to our understanding.

In order to reach a more precise description of the invariant dynamics of an IFS with probabilities, it is useful to introduce the notion of code space and of address of the points lying on the attractor.

Given a hyperbolic IFS $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$, the *code space* (Σ, ρ) associated with the IFS is defined to be the set of all semi-infinite sequences of N symbols $\{1, 2, \ldots, N\}$ with the metric ρ given by:

$$\rho(\omega, \sigma) = \sum_{n=1}^{\infty} \frac{|\omega_n - \sigma_n|}{(N+1)^n} \qquad \forall\, \omega, \sigma \in \Sigma \tag{18}$$

where ω_n, σ_n are respectevely the n-th symbols of the infinite strings ω and σ. It is fairly easy to show the existence of a surjection from the elements of the code space associated with the IFS to the points of the attractor F, that is a continuous and onto function $\phi : \Sigma \to F$ defined as the limit:

$$\phi(\sigma) = \lim_{n \to \infty} \phi(\sigma, n, \mathbf{x}_0) \tag{19}$$

where $\phi(\sigma, n, \mathbf{x}_0) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(\mathbf{x}_0)$. Then we call *address* of a point $\mathbf{x} \in F$ any member of the set:

$$\phi^{-1}(\mathbf{x}) = \{\sigma \in \Sigma : \phi(\sigma) = \mathbf{x}\} \tag{20}$$

where ϕ is defined in (19). When the IFS is totally disconnected[12] each point of its attractor possesses a unique address, that is ϕ is one-to-one and onto.

Let $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$ a totally disconnected hyperbolic IFS and F its attractor. The dynamical system $\{F, S\}$, where the transformation $S : F \to F$ is defined by:

$$S(\mathbf{x}) = w_n^{-1}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in w_n(F), \tag{21}$$

is called the *shift dynamical system* associated with the IFS. [Barnsley (1988)] proves that $\{F, S\}$ is topologically conjugate[13] to the dynamical

[12] An IFS is totally disconnected when each point of its attractor F is an isolated point. One can prove that F is totally disconnected if and only if:

$$w_i(F) \cap w_j(F) = \emptyset \qquad \forall i, j \in \{1, 2, \ldots n\},\, i \neq j$$

Moreover if the IFS is totally disconnected, then the transformation $w_n : F \to F$ is one to one, for all $n \in \{1, 2, \ldots, N\}$.

[13] Of course, the homeomorphism which provides this topological equivalence is the map $\phi : \Sigma \to F$ defined in (19).

system $\{\Sigma, T\}$, where Σ is the code space associated with the IFS and $T : \Sigma \to \Sigma$ (the *shift map*) is defined by:

$$T(\sigma_1\sigma_2\sigma_3\cdots) = \sigma_2\sigma_3\sigma_4\cdots \qquad \forall\ \sigma = \sigma_1\sigma_2\sigma_3\cdots \in \Sigma. \tag{22}$$

Consider now a given totally disconnected hyperbolic IFS with probabilities $\{\mathbf{X}; w_1, w_2, \ldots, w_N\ p_1, p_2, \ldots, p_N\}$. In section 3 we saw that the mechanism by which at each iteration it produces an orbit $\{\mathbf{x}_n\}_{n=0}^{\infty}$ consists of two stages: first choose an element i_n from the set $\{1, 2, \ldots, N\}$ according to the probabilities $\{p_1, p_2, \ldots, p_N\}$ and then calculate $\mathbf{x}_{t+1} = w_{i_n}(\mathbf{x}_t)$. That is, if $\mathbf{x}_0 \in F$ is the starting point and $\sigma \in \Sigma$ the corresponding address, we have:

$$\begin{aligned}
&\mathbf{x}_1 = w_{i_1}(\mathbf{x}_0) = \phi(i_1\sigma)\\
&\mathbf{x}_2 = w_{i_2}(\mathbf{x}_1) = w_{i_2} \circ w_{i_1}(\mathbf{x}_0) = \phi(i_2 i_1\sigma)\\
&\quad \cdots\cdots\\
&\mathbf{x}_n = w_{i_n} \circ w_{i_{n-1}} \circ \cdots \circ w_{i_1}(\mathbf{x}_0) = \phi(i_n i_{n-1}\cdots i_1\sigma)\\
&\quad \text{and so on ...}
\end{aligned}$$

At this point it is easy to show that if we repeatedly apply the transformation $S : F \to F$ (21) starting from the above $\mathbf{x}_n$ (or, equivalently, the transformation $T : \Sigma \to \Sigma$ (22) starting from the corresponding address $i_n i_{n-1} \ldots i_1\sigma$), we go along the same orbit but backward in time. Roughly speaking, the shift dynamical system associated with a totally disconnected IFS produces *backward dynamics*, that is sequences of states which coincide in reversed order with the single realizations of the IFS.

Keeping in mind that the shift map $T : \Sigma \to \Sigma$ is chaotic[14] [see

[14]It is not possible to give here a clear and satisfactory definition of chaotic system. Following the topological approach, a dynamical system $\{\mathbf{X}, f\}$ is said to be chaotic on $\mathbf{X}$ if it is a) *transitive*, b) *sensitive to initial conditions* (SDIC) and c) if the set of periodic orbits of f is dense in $\mathbf{X}$. Very briefly, $\{\mathbf{X}, f\}$ is transitive if for any pair of open subsets $U, V \subset \mathbf{X}$ there esists a number $k > 0$ such that $f^{(k)}(U) \cap V \neq \emptyset$, and it has SDIC if there exists $\delta > 0$ such that, for any $\mathbf{x} \in \mathbf{X}$ and any neighborhood N of $\mathbf{x}$, there exists $\mathbf{y} \in N$ and $n \geq 0$ such that $|f^{(n)}(\mathbf{x}) - f^{(n)}(\mathbf{y})| \geq \delta$. For a deep treatment see e.g. Guckenheimer and Holmes (1983), Wiggins (1988) and the introductory Devaney (1989).

Wiggins (1988)], one can state the following theorem which represents an important link between the topological structure of the attractor and the (backward) invariant dynamics of an IFS:

THEOREM 7. [Barnsley (1988)] *The shift dynamical system associated with a totally disconnected hyperbolic IFS of two or more transformations is chaotic.*

This result is undoubtedly surprising[15] and, at first sight, it may seem particularly attractive, expecially for those who are involved in economic policy. Therefore it is worth to point out what does it mean and what it doesn't. First of all the existence of a backward dynamical system allow us to answer, at least in principle, an important question: "Let $\mathbf{x}_{t+n}$ the production vector at time $t+n$, how should the system evolve from t to $t+n$ in order that it can reach exactly $\mathbf{x}_{t+n}$?". In fact, starting from the 'final condition' $\mathbf{x}_{t+n}$, the shift dynamical system (21) enables us to compute the time path of states that our system should follow during the n period, together with the sequence of shocks on the coefficients of the matrix A and the components of the vector demand $\mathbf{b}$ which should take place (and which have, in our model, an exogenous explanation). It is clear that this kind of information has no pratical relevance for a policy maker because, while in one way or another the production $\mathbf{x}$ can be controlled, the shocks are random and they cannot be chosen in advance. Of greater significance it would be to know the range of values within which the state $\mathbf{x}$ should fall at each period in order that the system could 'land' in a predefined neighbourhood of x_{t+n}. But at this point the chaoticity (and in particular the topological transitivity) of the backward dynamics comes into play in a crucial manner. In fact, if we consider in $t+n$ a ball of initial conditions centered in x_{t+n}, iteration after iteration it will remain no longer a ball or an ellipsoid, but it will rather become a Swiss cheese-like set which will scatter over the whole attractor F [see Fig. 1]. Therefore the hope that the shift dynamical system could tell us where to drive our system along the n periods vanishes.

[15]This is what Bartlett (1990) calls a 'curious example' of a stochastic process which, reversed in time, becomes transformed into a non-linear chaotic deterministic process. See also Degn (1982).

[Place Fig. 1 approximately here]

Unfortunately, a complete and satisfying characterization of the dynamical behavior of IFSs is currently not available. Moreover, even in the case of a totally disconnected F checking the condition for chaotic behavior requires at least a rough computation of the attractor and this is not always an easy task. Therefore, the most reasonable way to determine the behavior of our input-output dynamics for realistic parameter values is to undertake a simulative analysis starting from available estimates of production and demand coefficients. This is the purpose of the next section.

6. Simulations.

First case: no shocks on the production coefficients.

In order to clarify the different role played by the endogenous and exogenous components in our input-output model, we start considering the simple case with shocks on the demand side but no shocks on the coefficients of the matrix A. From the point of view of our model we are facing an IFS (with probabilities) whose affine maps w_i have all the same technological matrix A, that is $A_1 = \ldots = A_N = A$. If we call $\phi^D(A)$ their common singular value function, (15) becomes:

$$\lim_{r\to\infty}\left[\sum_{\mathbf{i}\in J_r}\phi^D(A^{\mathbf{i}})\right]^{\frac{1}{r}} = \lim_{r\to\infty}\left[\sum_{i=1}^{N^r}\prod_{j=1}^{r}\phi^D(A_j)\right]^{\frac{1}{r}} = N\cdot\phi^D(A) = 1 \quad (23)$$

and:

$$\dim F = \min\left\{n,\ m-1+\frac{\log N+\log\lambda_1+\ldots+\log\lambda_{m-1}}{\log\lambda_m^{-1}}\right\}. \quad (24)$$

The latter equation shows three significant facts: a) the dimension of the attractor may reach non integer values, so F may be a fractal set; b) $\dim F$ grows as $N = \text{card}\,\Theta$ grows but it saturates to n for $N \geq \frac{1}{\lambda_1\cdots\lambda_n}$; c) the shocks $\mathbf{b}_i$ on the demand vector do not affect in general $\dim F$ which is entirely determined by the technological coefficients.

We can elucidate the role of the demand shocks calculating the expected value $E(\mathbf{x}_t)$ and the variance $\sigma^2(\mathbf{x}_t)$ of the production vector, under the hyphotesis of IID shocks on the demand side[16]:

$$\begin{aligned} E(\mathbf{x}_t) &= A^t E(\mathbf{x}_0) + (I - A)^{-1}(I - A^t)E(\mathbf{b}) \\ \sigma^2(\mathbf{x}_t) &= A^{2t}\sigma_0^2 + (I - A)^{-1}(I - A^t)^2(I - A)^{-1}\sigma_{\mathbf{b}}^2. \end{aligned} \tag{25}$$

Of course, for $t \to \infty$, given that $s = |||A||| \leq 1$:

$$\begin{aligned} E(\mathbf{x}) &= (I - A)^{-1}E(\mathbf{b}) \\ \sigma^2(\mathbf{x}) &= [(I - A)^{-1}]^2\sigma_{\mathbf{b}}^2. \end{aligned} \tag{26}$$

As basis of our numerical simulations we take the 3×3 input–output matrix of the Italian economy for the 1982:

Matrix of the production coefficients of the Italian Economy†			
	Agriculture	*Industry*	*Services*
Agriculture	0.172	0.041	0.004
Industry	0.146	0.365	0.130
Services	0.061	0.110	0.170

†(Year: 1982. From: [Baranzini and Marangoni(1991)].)

together with the production and demand vectors for the same year:

$$\mathbf{x}_{1982} = \begin{bmatrix} 55310 \\ 663420 \\ 456950 \end{bmatrix} \qquad \mathbf{b}_{1982} = \begin{bmatrix} 16840 \\ 354000 \\ 302850 \end{bmatrix}$$

The calculation of the singular values of A gives:

$$\lambda_1 = 0.4623 \qquad \lambda_2 = 0.1466 \qquad \lambda_3 = 0.1106.$$

[16] Of course, if this hyphotesis does not hold, (26) becomes more complicate. In any case (14) does not depend on the probability distribution of $\mathbf{b}$.

We start considering a very simple case with only three arbitrary chosen shocks. Fig. 2 shows a simulated time evolution of the three sectors production levels. Fig. 3 reproduce the 3D plot of the attractor F. Fig. 4 shows the projections of the attractor on the three orthogonal planes x_1-x_2, x_1-x_3 and x_2-x_3. Following (24), the fractal dimension of F results equal to 1.17. In Fig. 5 we report an empirical estimate of the fractal dimension of the attractor obtained via the box–counting method [see Barnsley (1988)]; our findings substantially agree with the theoretical estimate.

We have also performed some simulations of the case of an uncountable number of shocks. Figure 6 shows the time evolution in this case. Figs. 7 and 8 illustrate, respectively, the 3D representation of the attractor and its projections. As remarked above, note that the attractor tends to fill the whole three–dimensional space (its dimension thus approaches 3).

Second case: shocks on the production coefficients.

This case it at the moment too complex to allow for analytic considerations. For this reason, we limit ourselves to a few exploratory simulations, leaving a more thorough analysis for future research.

Two particular situations have been considered. In the first place we have dealt with the instance of five shocks acting on the coefficient matrix only. The resulting attractor is shown in Fig. 9; in Fig. 10 we report the corresponding box–counting dimension estimation. Lacking of analytical results for this case, we cannot compare our findings with a theoretical estimation.

We have then proceeded to the instance where the shocks simultaneously act on the coefficient matrix and the demand vector. The attractor and the empirical estimation of the fractal dimension are reported in Figs. 11 and 12, respectively. Comparing these results with those obtained when the shocks only act on the matrix of coefficients, we basically find the same value for the fractal dimension. This is what we expected on the basis of the theory, as pointed out in section 3.

From our simulations an interesting regularity emerges: in all cases considered, independently of whether the shocks act on the matrix of coefficients or the demand vector or both, the corresponding attractors are made up by a number of ‘blocks’ equal to the number of shocks considered (in particular, in the case of an uncountable number of shocks, the attractor is a dense cloud that tends to fill the whole space). This is not surprising: each shock can be interpreted as a 'displacement' of the same basic structure built in the IFS 'code' given by the technical coefficients [as illustrated e.g. in the discussion of the 'Collage Theorem' by Barnsley (1988)].

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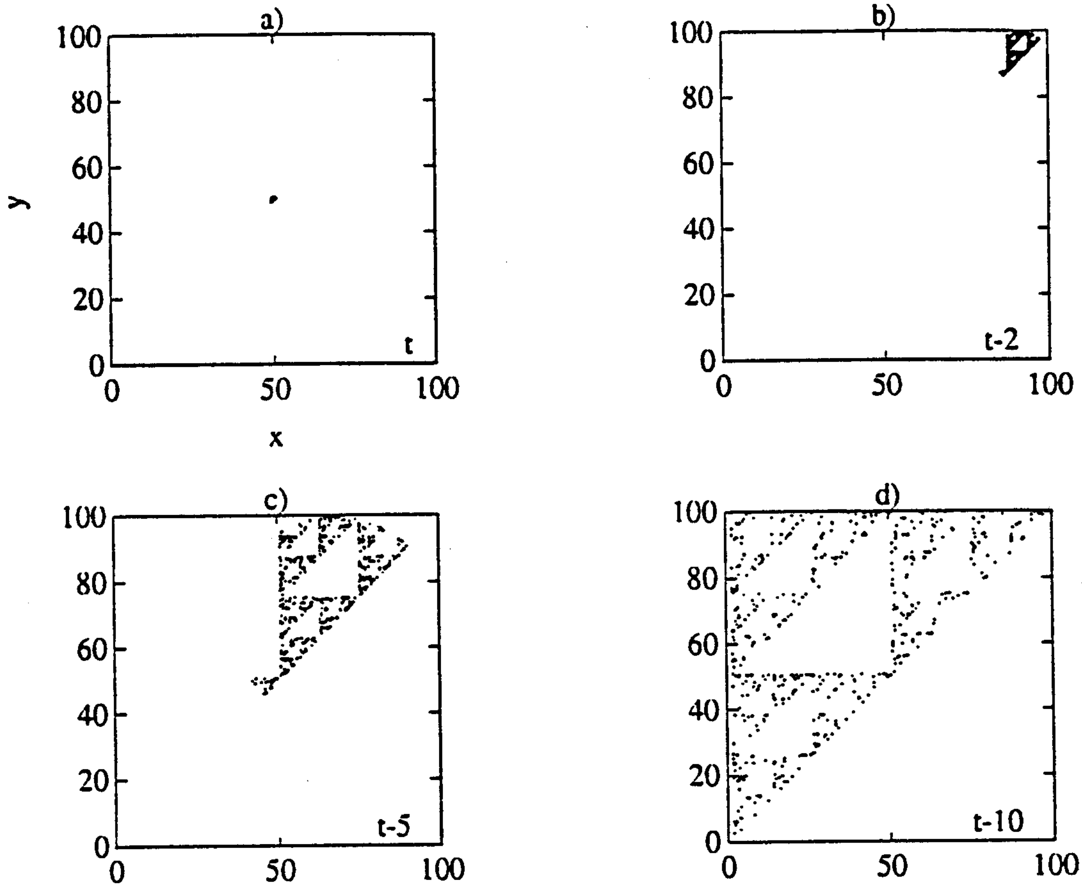

Fig. 1. In a) we consider a ball of initial conditions at time t of radius 1; b), c) and d) show how this set evolves after 2, 5 and 10 backward iterations under the action of the chaotic shift dynamical system (21). The quasi-totally disconnected (just– touching) IFS used in these simulations is:

$$w_1 = A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$w_2 = A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 50 \end{bmatrix} \qquad \text{where: } A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$w_3 = A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

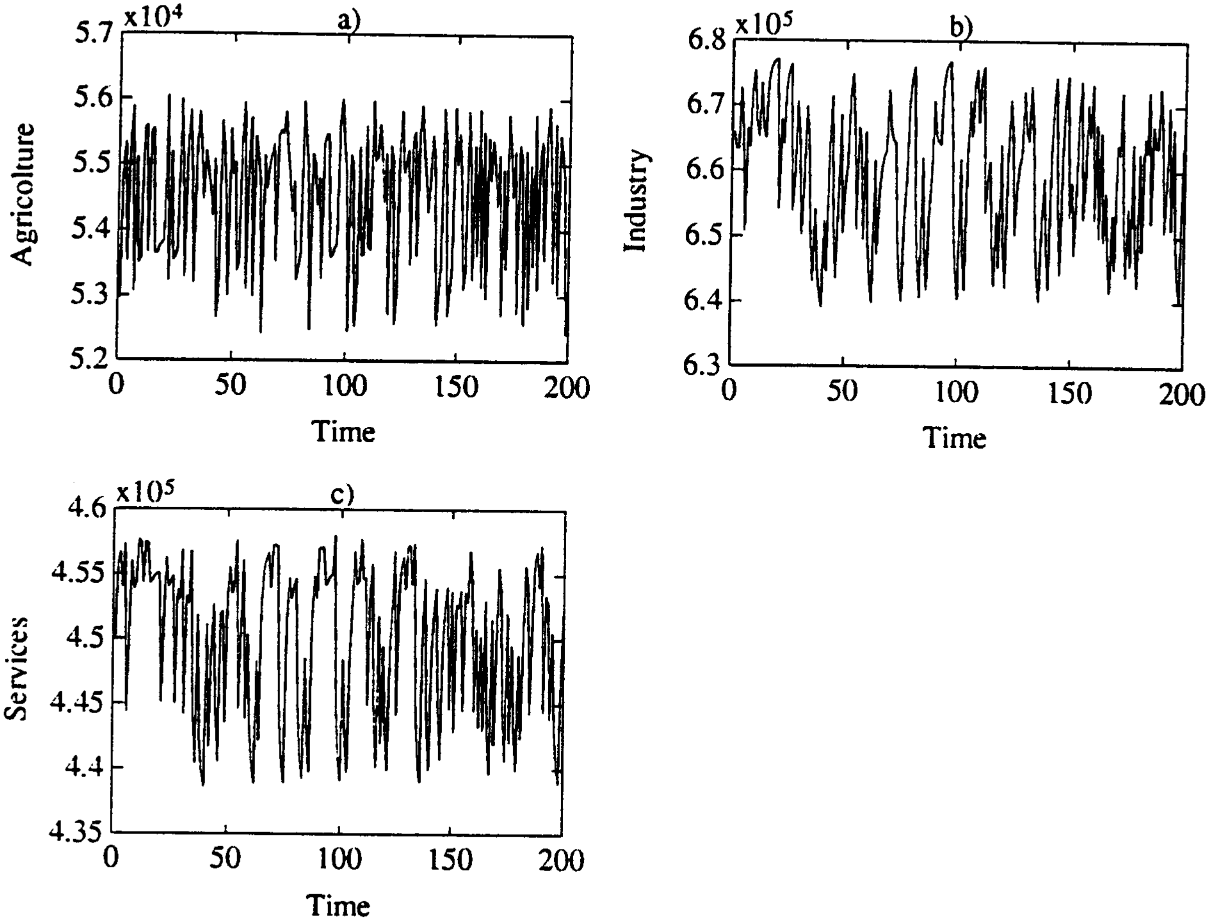

Fig. 2. 3-shocks case: time evolution of the production levels in the three sectors: a) Agricolture, b) Industry, c) Services.

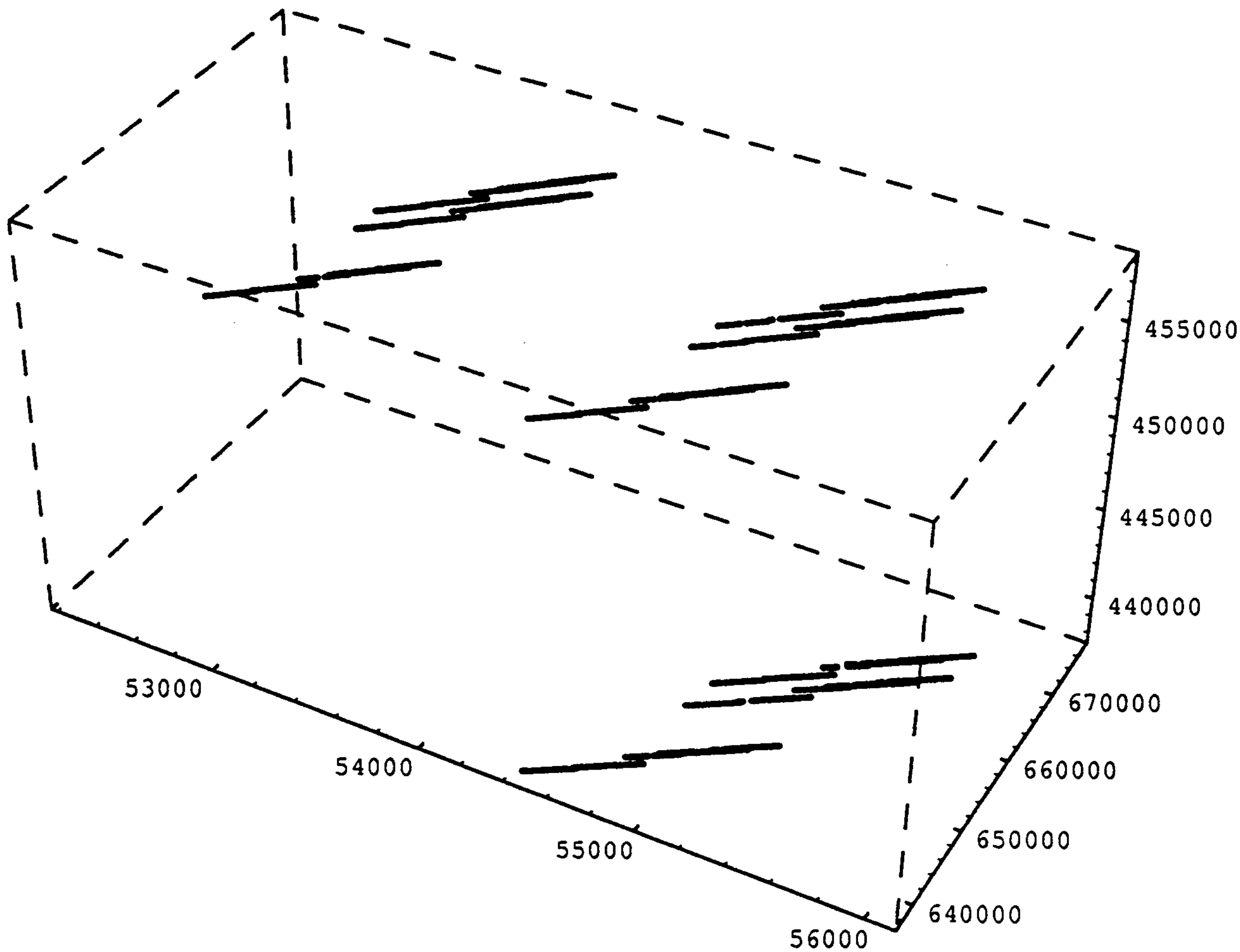

Fig. 3. 3-shocks case: 3D representation of the IFS attractor.

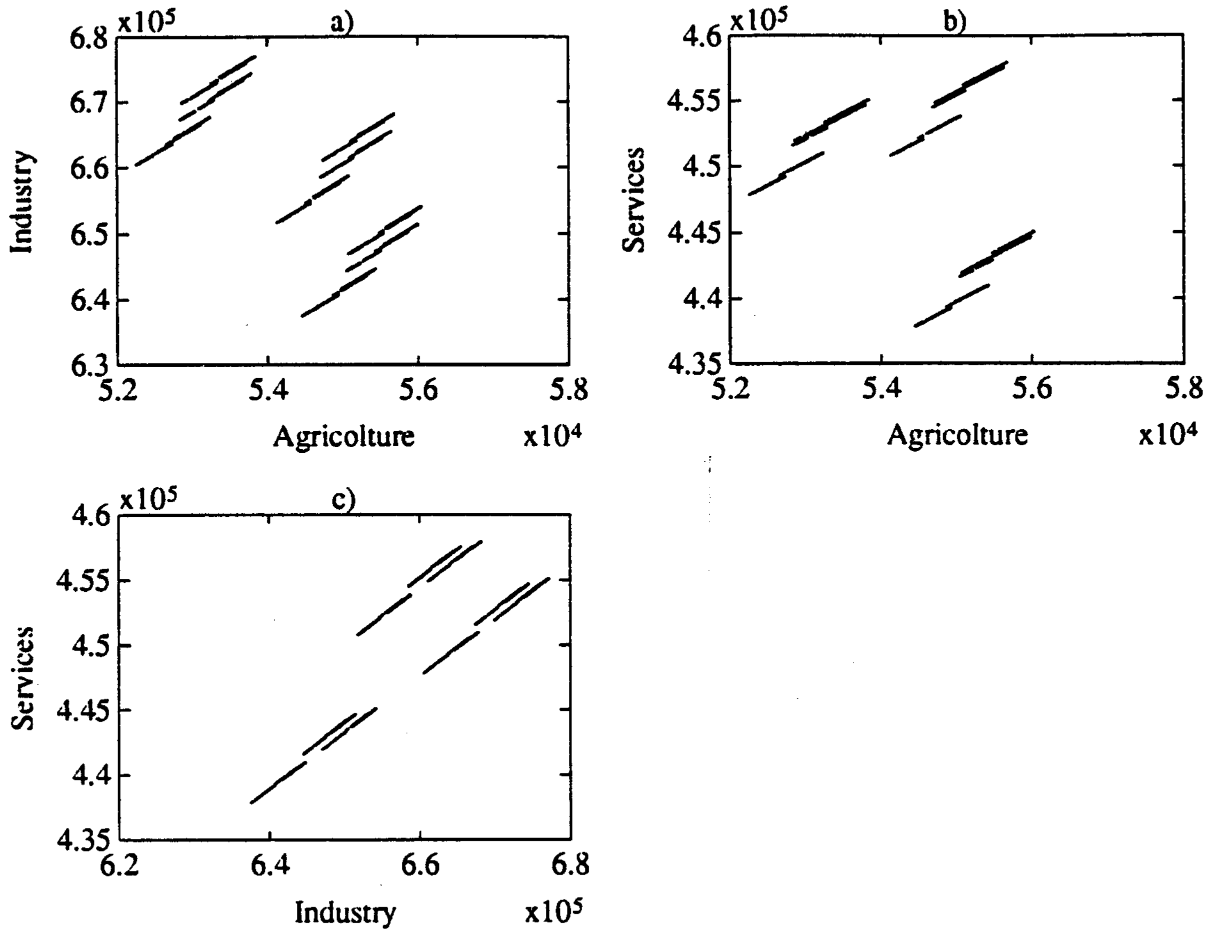

Fig. 4. 3-shocks case: orthogonal projections of the attractor on the a) x_1-x_2, b) x_1- x_3, and c) x_2-x_3 planes.

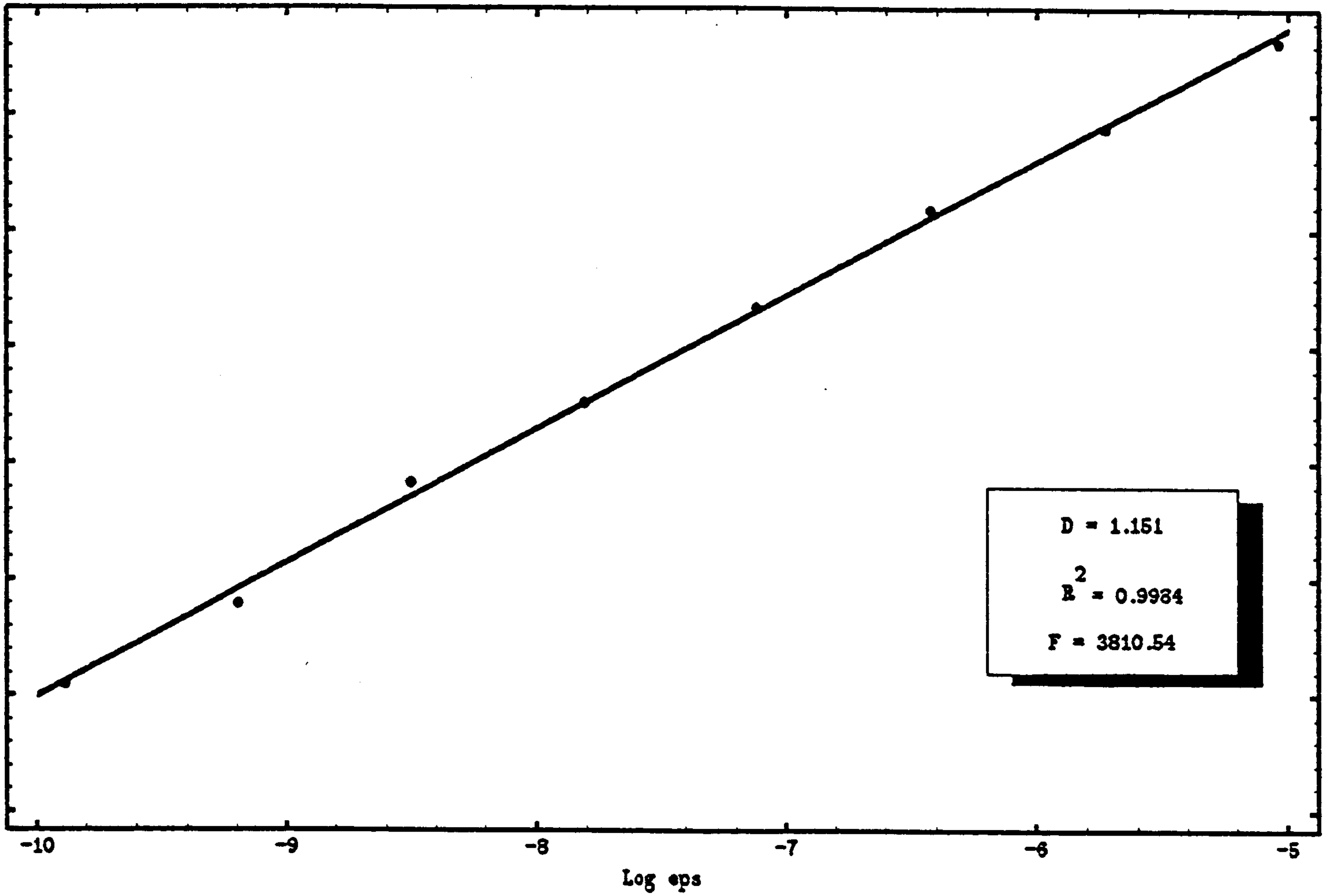

Fig. 5. 3-shocks case: empirical estimation of the fractal dimension.

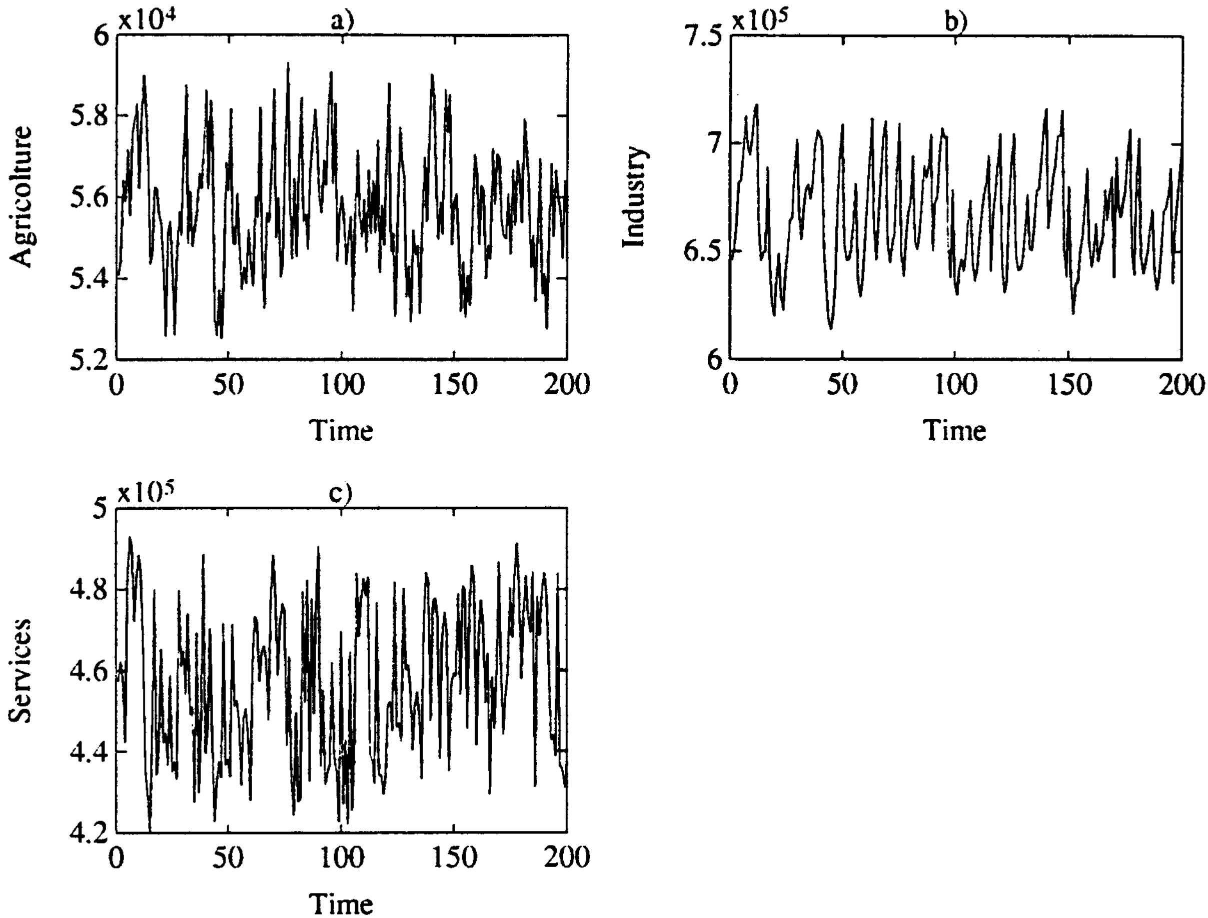

Fig. 6. Uncountably many shocks case: time evolution of the production levels in the three sectors: a) Agriculture, b) Industry, c) Services.

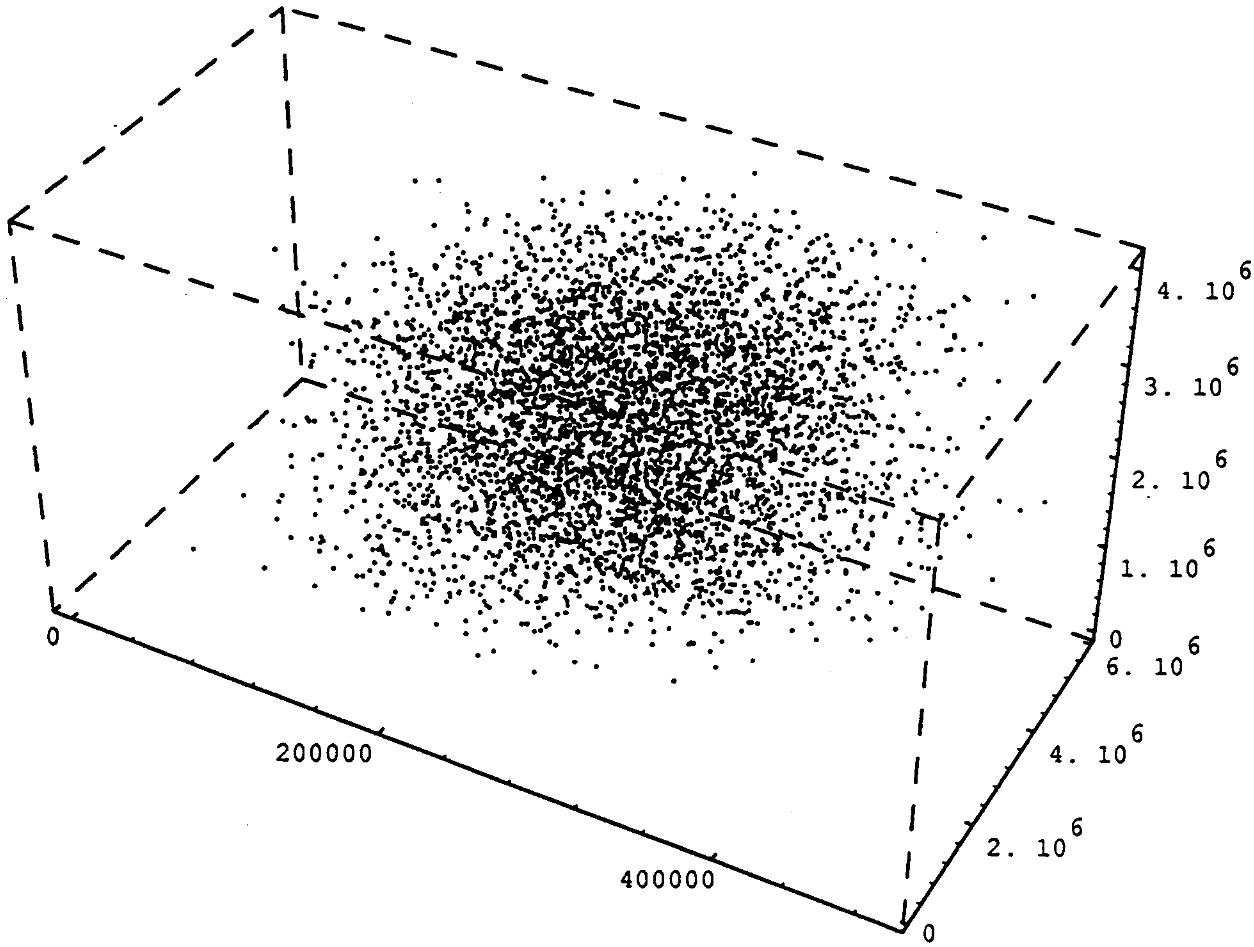

Fig. 7. Uncountably many shocks case: 3D representation of the IFS attractor.

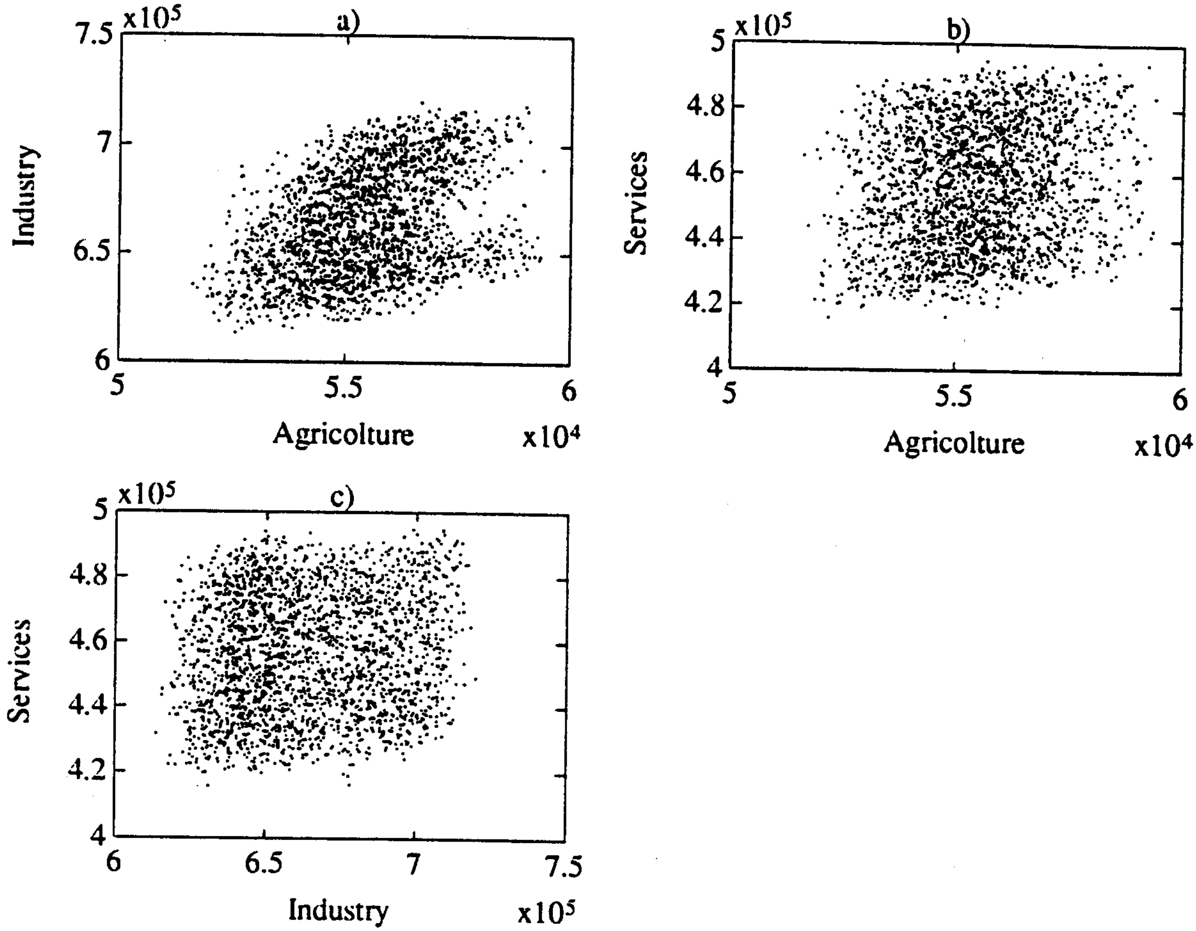

Fig. 8. Uncountably many shocks case: orthogonal projections of the attractor on the a) x_1-x_2, b) x_1- x_3, and c) x_2-x_3 planes.

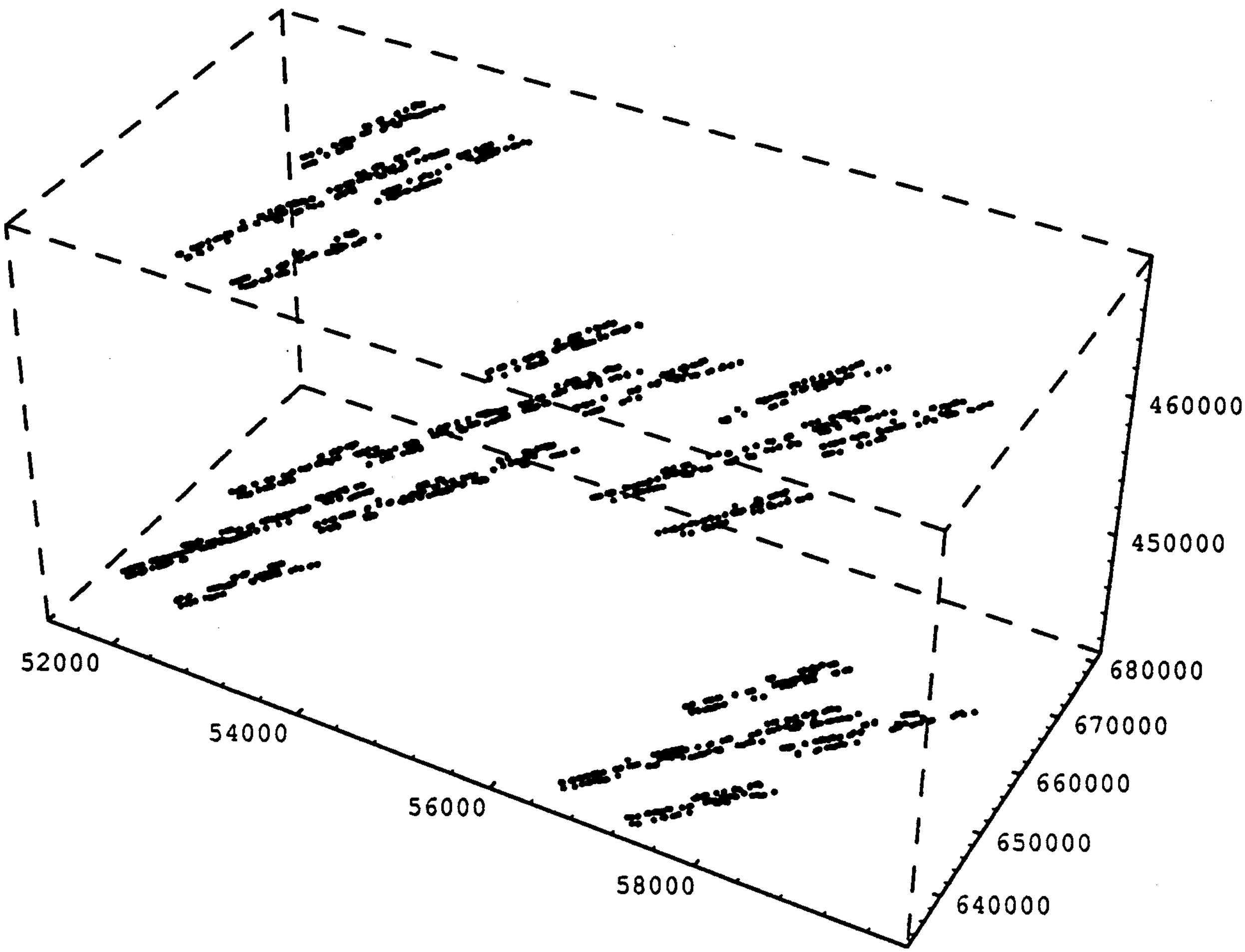

Fig. 9. 5-shocks case (on the matrix of coefficients only): 3D representation of the IFS attractor.

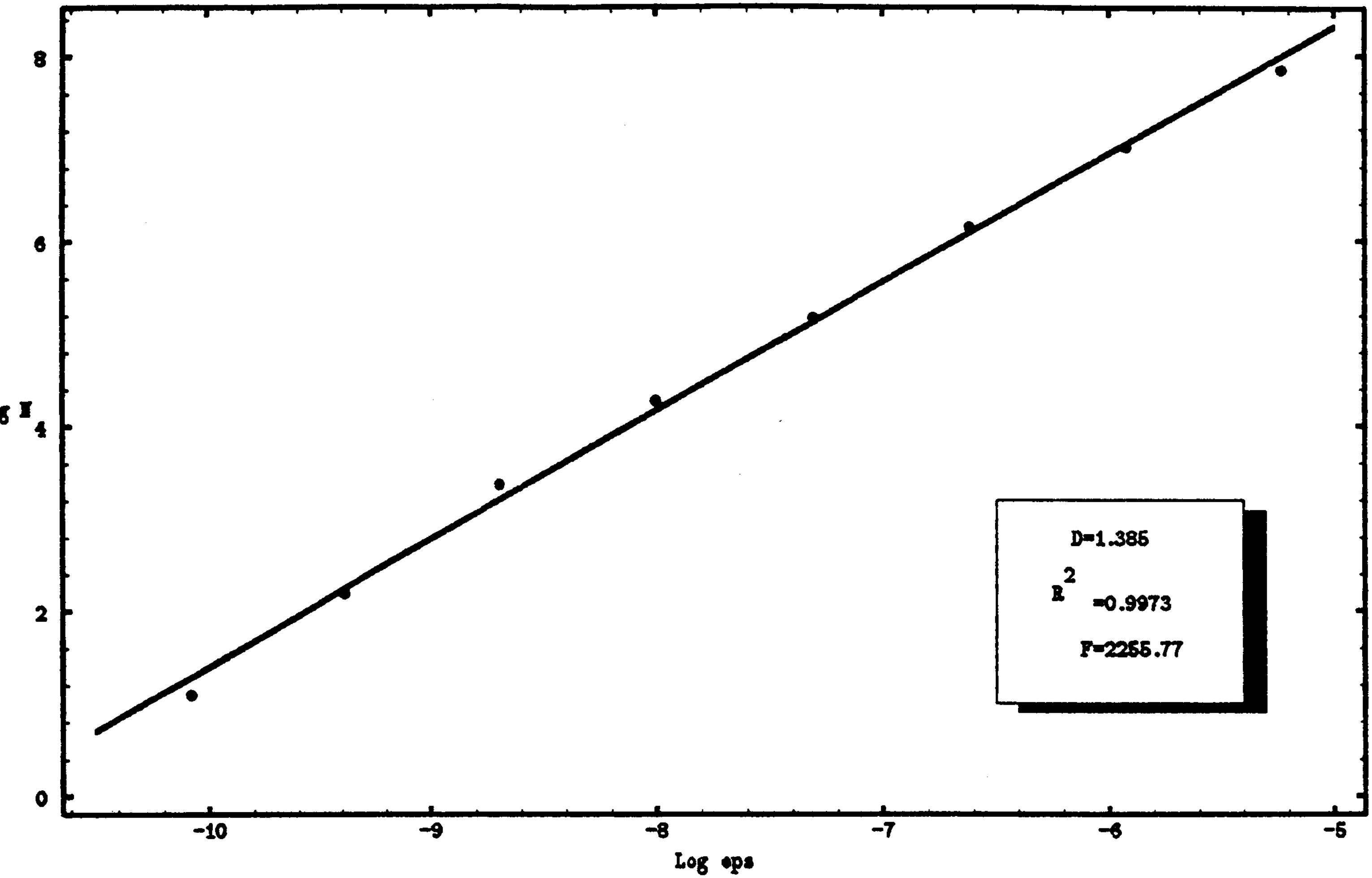

Fig. 10. 5-shocks case (on the matrix of coefficients only): empirical estimation of the fractal dimension.

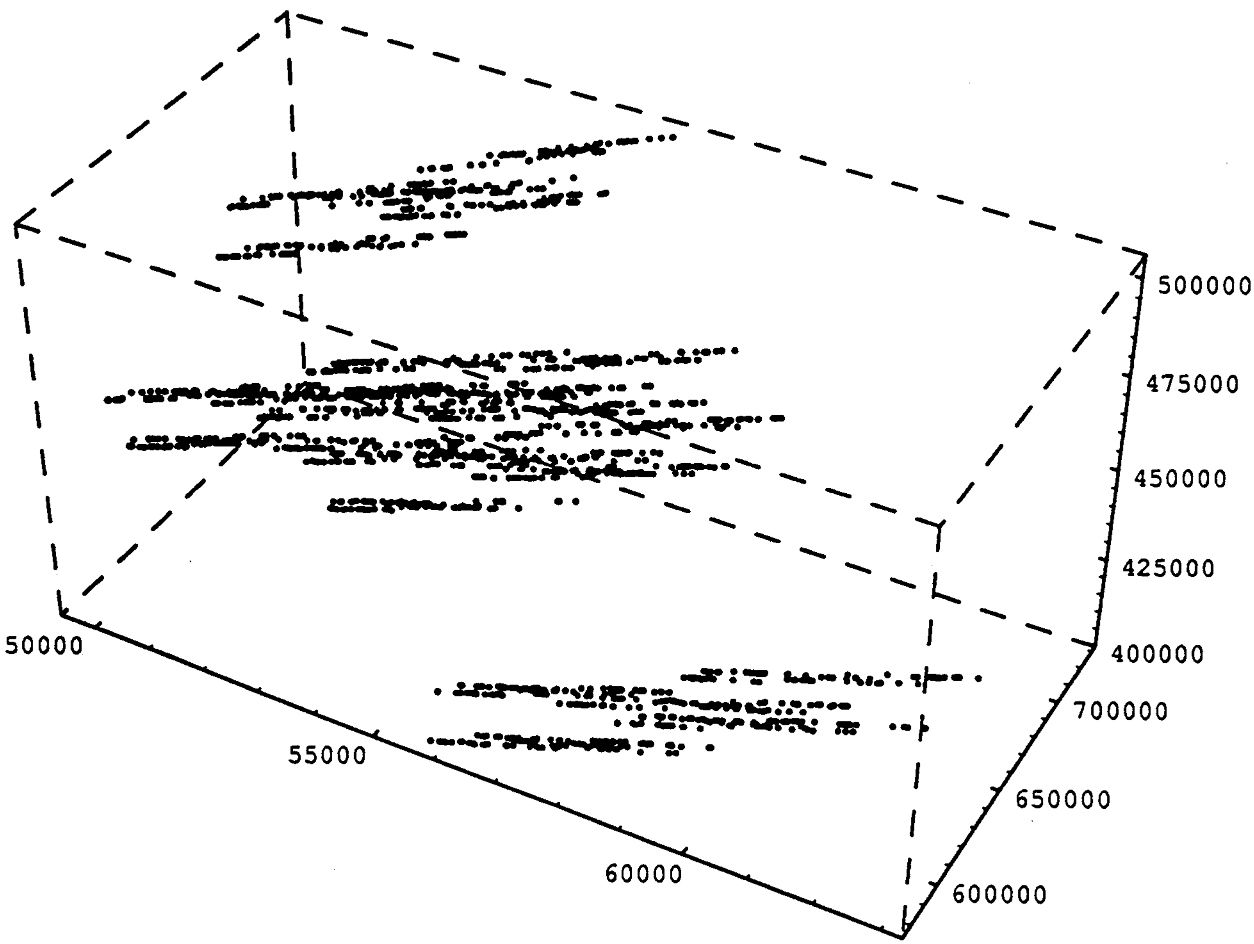

Fig. 11. 5-shocks case (both on the matrix of coefficients and the demand vector): 3D representation of the IFS attractor.

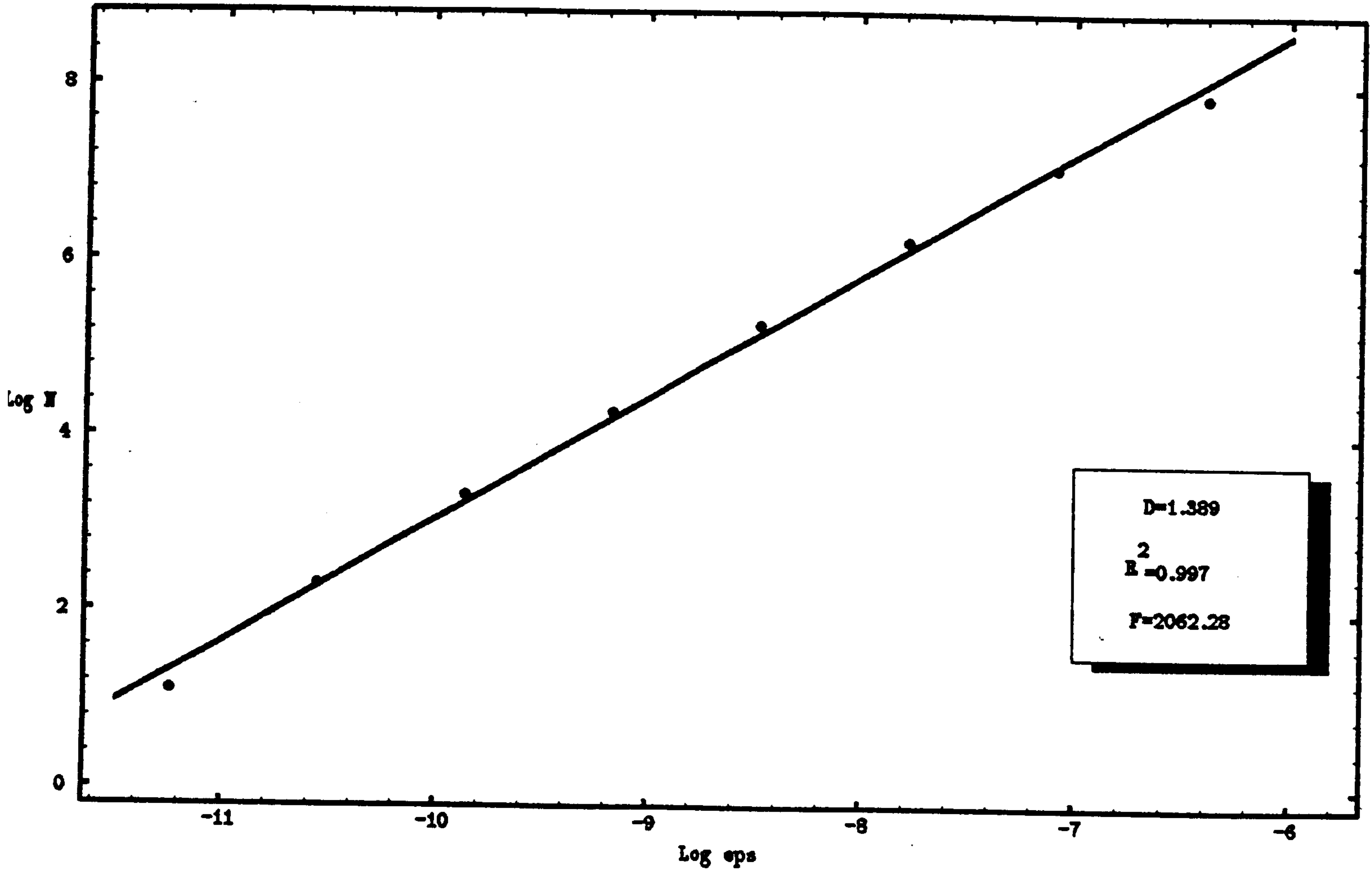

Fig. 12. 5-shocks case (both on the matrix of coefficients and the demand vector): empirical estimation of the fractal dimension.